On stability and sensitivity of constraint and variational systems

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Problem formulation

Given a closed-graph multifunction $M : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^l$, the associated *implicit multifunction* $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is defined by

$$S(p) := \{x \in \mathbb{R}^m | 0 \in M(p, x)\}. \tag{1}$$

Our main aim is analysis of Lipschitzian properties of S around a given *reference point* $(\bar{p}, \bar{x}) \in \operatorname{gph} S$. In particular, we will focus on the so-called Aubin property.

Special cases:

1) *parameterized constraint systems*

$$M(p, x) = G(p, x) + \Lambda, \tag{2}$$

where $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^l$ is single-valued and $\Lambda \subset \mathbb{R}^l$ is closed.

2) *parameterized variational systems*

$$M(p, x) = H(p, x) + Q(x), \tag{3}$$

where H is like in (i) and $Q : \mathbb{R}^m \rightrightarrows \mathbb{R}^l$ is closed-valued. Typically, $l = m$ and $Q(\cdot) = N_\Gamma(\cdot)$ with a closed set $\Gamma \subset \mathbb{R}^m$. Case (3) then leads to a *parameterized variational inequality/generalized equation*.

Applications

1) *Post-optimal analysis of parameterized equilibria*: Having an equilibrium computed for a given set of parameters (problem data), one tries to detect whether, roughly speaking, a small change of a some parameters (uncertain data) leads to a proportional change of the respective equilibrium.
2) *Solution of MPECs/EPECs*: In a hierarchical bilevel game the lower-level players typically compute a non-cooperative equilibrium, parameterized by the strategy(ies) of the upper-level player(s). The local stability analysis of this mapping is essential in computing optimal strategies of the upper-level player(s). Such a situation arises, e.g., in deregulated electricity markets of in optimal design of some mechanical structures.

(i) Selected tools of variational analysis;
(ii) Basic Lipschitzian stability notions from the theory of multifunctions;
(iii) Existing criteria for the Aubin property;
(iv) Aubin property of implicit multifunctions;
(v) Testing the Aubin property of parameterized constraint and variational systems;
(vi) Variational systems with a special constraint structure;
(vii) Conclusion.

Definition

Given a closed set $A \subset \mathbb{R}^n$ and $\bar{x} \in A$, we define

(i) the *tangent (Bouligand) cone* to A at $\bar{x}$ by

$$T_A(\bar{x}) := \{h \in \mathbb{R}^n | \exists h_i \to h, \vartheta_i \searrow 0 : \bar{x} + \vartheta_i h_i \in A \forall i\};$$

(ii) the *regular (Fréchet) normal cone* to A at $\bar{x}$ by

$$\widehat{N}_A(\bar{x}) := (T_A(\bar{x}))^\circ;$$

(iii) the *limiting (Mordukhovich) normal cone* to A at $\bar{x}$ by

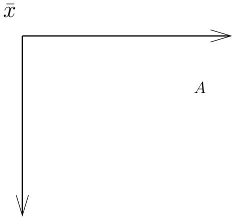
$$N_A(\bar{x}) := \{\xi \in \mathbb{R}^n | \exists x_i \xrightarrow{A} \bar{x}, \xi_i \to \xi : \xi_i \in \widehat{N}_A(x_i) \forall i\}.$$

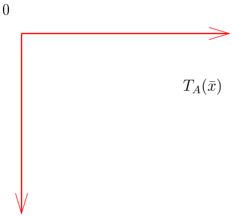
(iv) Finally, given a direction $h \in \mathbb{R}^n$, the cone

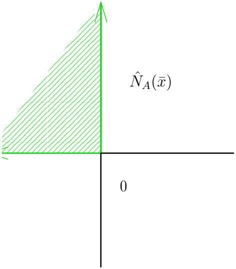
$$N_A(\bar{x}; h) := \{\xi \in \mathbb{R}^n | \exists h_i \to h, \vartheta_i \searrow 0, \xi_i \to \xi : \xi_i \in \widehat{N}_A(\bar{x} + \vartheta_i h_i) \forall i\}$$

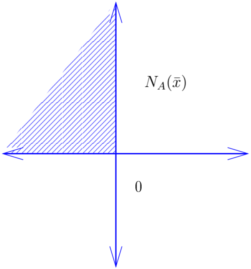
is called the *directional limiting normal cone* to A at $\bar{x}$ in the direction h.

Ad (i) Example

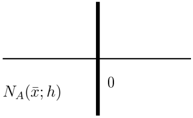




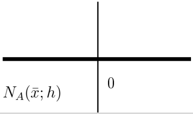




For $h \in \mathbb{R}_+ \times \{0\}$



For $h \in \{0\} \times \mathbb{R}_+$



Definition

Consider a point $(\bar{u}, \bar{v}) \in \mathrm{Gr}\, F$. Then

(i) the multifunction $DF(\bar{u}, \bar{v}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^l$, defined by

$$DF(\bar{u}, \bar{v})(h) := \{k \in \mathbb{R}^l | (h, k) \in T_{\mathrm{gph}\, F}(\bar{u}, \bar{v})\}, h \in \mathbb{R}^n,$$

is called the *graphical derivative* of F at $(\bar{u}, \bar{v})$;

(ii) the multifunction $\hat{D}^* F(\bar{u}, \bar{v}) : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$, defined by

$$\hat{D}^* F(\bar{u}, \bar{v})(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in \hat{N}_{\mathrm{gph}\, F}(\bar{u}, \bar{v})\}, v^* \in \mathbb{R}^l,$$

is called the *regular (Fréchet) coderivative* of F at $(\bar{u}, \bar{v})$.

(iii) the multifunction $D^* F(\bar{u}, \bar{v}) : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$, defined by

$$D^* F(\bar{u}, \bar{v})(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in N_{\mathrm{gph}\, F}(\bar{u}, \bar{v})\}, v^* \in \mathbb{R}^l,$$

is called the *limiting (Mordukhovich) coderivative* of F at $(\bar{u}, \bar{v})$.

(iv) Finally, given a pair of directions $(h, k) \in \mathbb{R}^n \times \mathbb{R}^l$, the multifunction $D^* F((\bar{u}, \bar{v}); (h, k)) : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$, defined by

$$D^* F((\bar{u}, \bar{v}); (h, k))(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in N_{\mathrm{gph}\, F}((\bar{u}, \bar{v}); (h, k))\}, v^* \in \mathbb{R}^l, \tag{4}$$

is called the *directional limiting coderivative* of F at $(\bar{u}, \bar{v})$ in direction (h, k).

Ad (ii) Basic Lipschitzian stability notions

Consider a multifunction $S : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$ and a point $(\bar{v}, \bar{u}) \in \mathrm{gph}\, S$.

1) We say that S has a *single valued Lipschitzian localization* around $(\bar{v}, \bar{u})$, provided $\exists$ neighborhoods V, U of $\bar{v}, \bar{u}$, respectively, and Lipschitzian function $\sigma : V \to \mathbb{R}^n$ such that
$$\bar{u} = \sigma(\bar{v}) \text{ and } S(v) \cap U = \{\sigma(v)\} \text{ for all } v \in V.$$
2) S has the *Aubin property* around $(\bar{v}, \bar{u})$, provided $\exists$ neighborhoods V, U of $\bar{v}, \bar{u}$, respectively, and a constant $\kappa > 0$ such that
$$S(v') \cap U \subset S(v) + \kappa \| v - v' \| \mathbb{B}_{\mathbb{R}^l} \text{ for all } v, v' \in V.$$
3) S is *isolatedly calm* at $(\bar{v}, \bar{u})$, provided $\exists$ neighborhoods V, U of $\bar{v}, \bar{u}$, respectively, and a constant $\kappa > 0$ such that
$$S(v) \cap U \subset \{\bar{u}\} + \kappa \| v - \bar{v} \| \mathbb{B}_{\mathbb{R}^l} \text{ for all } v \in V.$$
4) S is *calm* at $(\bar{v}, \bar{u})$, provided $\exists$ neighborhoods, V, U of $\bar{v}, \bar{u}$, respectively, and a constant $\kappa > 0$ such that
$$S(v) \cap U \subset S(\bar{v}) + \kappa \| v - \bar{v} \| \mathbb{B}_{\mathbb{R}^l} \text{ for all } v \in V.$$

Ad (ii) Basic Lipschitzian stability notions

S has a single-valued Lipschitzian localization around $(\bar{v}, \bar{u})$	$\Longrightarrow$	S is isolatedly calm at $(\bar{v}, \bar{u})$
$\Downarrow$		$\Downarrow$
S has the Aubin property around $(\bar{v}, \bar{u})$	$\Longrightarrow$	S is calm at $(\bar{v}, \bar{u})$

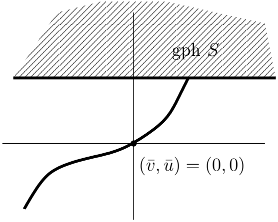
It is well-known that S has the Aubin property around $(\bar{v}, \bar{u})$ iff $F := S^{-1}$ is *metrically regular* at $(\bar{u}, \bar{v})$, i.e., $\exists$ neighborhoods U, V of $\bar{u}, \bar{v}$, respectively, and a constant $\kappa > 0$ such that

$$d(u, F^{-1}(v)) \leq \kappa\, d(v, F(u)) \text{ for all } u \in U, v \in V.$$

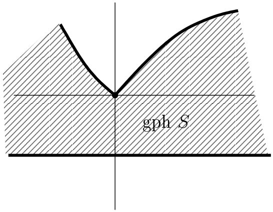
Likewise S is calm at $(\bar{v}, \bar{u})$ iff $F := S^{-1}$ is *metrically subregular* at $(\bar{u}, \bar{v})$, i.e., $\exists$ a neighborhood U of $\bar{u}$ and a constant $\kappa > 0$ such that

$$d(u, F^{-1}(\bar{v})) \leq \kappa\, d(\bar{v}, F(u)) \text{ for all } u \in U.$$

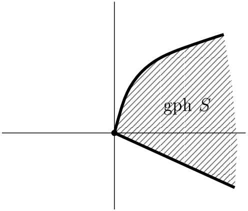
Ad (ii) Basic Lipschitzian stability notions



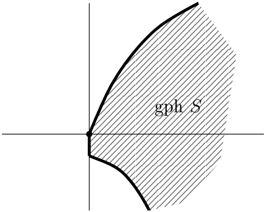
Ad 1) $D_*S(\bar{v}, \bar{u})(0) = \{0\}$



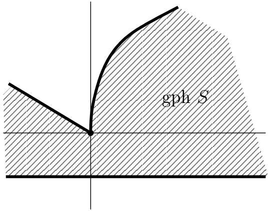
Ad 2) $D^*S(\bar{v}, \bar{u})(0) = \{0\}$



Ad 3) $DS(\bar{v}, \bar{u})(0) = \{0\}$



Ad 4)



Non-Lipschitzian behavior near $(\bar{v}, \bar{u})$

1) Via Mordukhovich criterion [M92]. This characterization, combined with the coderivative chain rule from [HJO02], yields

Theorem 1

Assume that

- M is metrically subregular at $(\bar{p}, \bar{x}, 0)$;
- The implication

$$(q^*, 0) \in D^* M(\bar{p}, \bar{x}, 0)(b^*) \Rightarrow q^* = 0. \tag{5}$$

 holds true.

Then S has the Aubin property around $(\bar{p}, \bar{x})$.

Ad (iii) Existing criteria for the Aubin property

2) [DQZ06] Assume that M has the Aubin property with respect to p uniformly in x, i.e., there exist a constant $\alpha > 0$ and neighborhoods O of 0, P of $\bar{p}$ and W of $\bar{x}$ such that

$$M(p', x) \cap O \subset M(p, x) + \alpha \|p' - p\| \, \mathbb{B}_{\mathbb{R}^l} \text{ for all } p, p' \in P, x \in W. \tag{6}$$

Then, with

$$M_{\bar{p}}(x) := M(\bar{p}, x),$$

one has the implication:

$$\left.\begin{array}{l} M_{\bar{p}} \text{ is metrically regular} \\ \text{around } (\bar{x}, 0) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} S \text{ has the Aubin property} \\ \text{around } (\bar{p}, \bar{x}) \end{array}\right. \tag{7}$$

Ad (iii) Discussion

The above two criteria are not directly comparable. Consider therefore the special cases (2), (3) and assume that G, H are continuously differentiable. Then (6) is automatically fulfilled but condition (7) is more restrictive than (5). On the other hand, (5) is applicable only under the metric subregularity of M at $(\bar{p}, \bar{x}, 0)$. If $\nabla_p G(\bar{p}, \bar{x}), \nabla_p H(\bar{p}, \bar{x})$ are surjective (ample perturbations), then (5) ensures at the same time the metric subregularity of M at $(\bar{p}, \bar{x}, 0)$, both above criteria coincide and amount to a *characterization* of the Aubin property of S around $(\bar{p}, \bar{x})$. Otherwise, however, both of them may be far from necessity and hence not quite satisfactory.

Ad (iv) Aubin property of implicit multifunctions

The new approach relies on the possibility to express the Mordukhovich criterion in terms of the directional limiting coderivatives and on the fact that, for a multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^l$ and $(\bar{u}, \bar{v}) \in \operatorname{gph} F$, $D^*F((\bar{u}, \bar{v}); (h, k))(a) = \emptyset$ for all a whenever $(h, k) \notin T_{\operatorname{gph} F}(\bar{u}, \bar{v})$.

Theorem 2.

Assume that

- M is metrically subregular at $(\bar{p}, \bar{x}, 0)$;
- $$\{u | 0 \in DM(\bar{p}, \bar{x}, 0)(v, u)\} \neq \emptyset \text{ for all } v \in \mathbb{R}^n. \tag{8}$$
- For every nonzero $(v, u) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $0 \in DM(\bar{p}, \bar{x}, 0)(v, u)$ the implication
$$(q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0); (v, u, 0))(b^*) \Rightarrow q^* = 0. \tag{9}$$
holds true.

Then S has the Aubin property around $(\bar{p}, \bar{x})$ and $DS(\bar{x}, \bar{y})(\cdot)$ admits the representation

$$DS(\bar{p}, \bar{x})(v) = \{u | 0 \in DM(\bar{p}, \bar{x}, 0)(v, u)\}, \; v \in \mathbb{R}^n. \tag{10}$$

Ad (iv) Aubin property of implicit multifunctions

Corollary.

If, in addition,

$$0 \in DM(\bar{p}, \bar{x}, 0)(0, u) \Rightarrow u = 0,$$

then S is isolatedly calm at $(\bar{p}, \bar{x})$.

Remarks.

Equality (10) means that the graphical derivative of S at $(\bar{p}, \bar{x})$ is implicitly given by the graphical derivative of M at $(\bar{p}, \bar{x}, 0)$. This directly generalizes the classical formula for the derivative of the implicit functions (U. Dini, 1877).

Since condition (8) is necessary for S to have the Aubin property and the directional limiting coderivatives are typically much smaller than the standard ones, the conditions of Theorem 2 are typically less restrictive than the conditions of Theorem 1.

Ad (iv) If we cannot establish the metric subregularity of M?

Theorem 3.

Let us omit the first assumption of Theorem 2 and strengthen the implication (9) to

$$(q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0); (v, u, 0))(b^*) \Rightarrow q^* = 0, b^* = 0. \quad (11)$$

Then the assertions of Theorem 2 remain valid.

This follows from the fact that (11) ensures at the same time the metric subregularity of M at $(\bar{p}, \bar{x}, 0)$ due to FOSCMS (see [GO15]). The classical counterpart of Theorem 3 from [M1, Section 4.3] reads:

Theorem 4.

Assume that

$$(q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0)(b^*) \Rightarrow q^* = 0, b^* = 0.$$

Then S has the Aubin property around $(\bar{p}, \bar{x})$.

Ad (v) Application to parameterized constraint systems

Consider the special case (2), where

$$M(p,x) = -G(p,x) + \Lambda.$$

Theorem 5.

Assume that G is continuously differentiable and

- M is metrically subregular at $(\bar{p}, \bar{x}, 0)$;
- $\{u | \nabla_p G(\bar{p},\bar{x})v + \nabla_x G(\bar{p},\bar{x})u \in T_\Lambda(G(\bar{p},\bar{x}))\} \neq \emptyset$ for all $v \in \mathbb{R}^n$;
- For every nonzero $(v,u) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\nabla_p G(\bar{p},\bar{x})v + \nabla_x G(\bar{p},\bar{x})u \in T_\Lambda(G(\bar{p},\bar{x}))$$

the implication

$$\left.\begin{array}{l} 0 = \nabla_x G(\bar{p},\bar{x})^T b^* \\ b^* \in N_\Lambda(G(\bar{p},\bar{x}); \nabla_p G(\bar{p},\bar{x})v + \nabla_x G(\bar{p},\bar{x})u) \end{array}\right\} \Rightarrow b^* \in \ker(\nabla_p G(\bar{p},\bar{x}))^T \quad (12)$$

holds true.

Then S has the Aubin property around $(\bar{p},\bar{x})$ and

$$DS(\bar{p},\bar{x})(v) = \{u | \nabla_p G(\bar{p},\bar{x})v + \nabla_x G(\bar{p},\bar{x})u \in T_\Lambda(G(\bar{p},\bar{x}))\}.$$

Ad (v) If we cannot establish the metric subregularity of M?

Theorem 6.

Let us omit the first assumption of Theorem 5 and strengthen the implication (12) to

$$\left.\begin{array}{l} 0 = \nabla_x G(\bar{p},\bar{x})^T b^* \\ b^* \in N_\Lambda(G(\bar{p},\bar{x}); \nabla_p G(\bar{p},\bar{x})v + \nabla_x G(\bar{p},\bar{x})u) \end{array}\right\} \Rightarrow b^* = 0. \tag{13}$$

Then the assertions of Theorem 5 remain valid.

Consider the special case (3), where $M(p,x) = H(p,x) + N_\Gamma(x)$.

Theorem 7.

Let $l = m$, H be continuously differentiable and $\Gamma \subset \mathbb{R}^m$ be convex and closed. Assume that

- M is metrically subregular at $(\bar{p}, \bar{x}, 0)$;
- $\{u \vert 0 \in \nabla_p H(\bar{p},\bar{x})v + \nabla_x H(\bar{p},\bar{x})u + DN_\Gamma(\bar{x}, -H(\bar{p},\bar{x}))(u)\} \neq \emptyset$ for all $v \in \mathbb{R}^n$;
- For every nonzero $(v,u) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$0 \in \nabla_p H(\bar{p},\bar{x})v + \nabla_x H(\bar{p},\bar{x})u + DN_\Gamma(\bar{x}, -H(\bar{p},\bar{x}))(u) \tag{14}$$

the implication

$$\left.\begin{array}{l} 0 \in (\nabla_x H(\bar{p},\bar{x}))^T b^* + \\ D^* N_\Gamma((\bar{x}, -H(\bar{p},\bar{x})); (u, -\nabla_p H(\bar{p},\bar{x}))v - \nabla_x H(\bar{p},\bar{x})u))(b^*) \end{array}\right\} \Rightarrow \tag{15}$$
$$\Rightarrow b^* \in \ker(\nabla_p H(\bar{p},\bar{x}))^T$$

holds true.

Then S has the Aubin property around $(\bar{p},\bar{x})$ and

$$DS(\bar{p},\bar{x})(v) = \{u \vert 0 \in \nabla_p H(\bar{p},\bar{x})v + \nabla_x H(\bar{p},\bar{x})u + DN_\Gamma(\bar{x}, -H(\bar{p},\bar{x}))(u)\}.$$

Remark. If Γ is polyhedral, then

$$DN_\Gamma(\bar{x}, -H(\bar{p}, \bar{x}))(u) = N_K(u),$$

where $K := \mathcal{K}_\Gamma(\bar{x}, H(\bar{p}, \bar{x})) = T_\Gamma(\bar{x}) \cap [H(\bar{p}, \bar{x})]^\perp$ (critical cone to Γ at $\bar{x}$ with respect to $H(\bar{p}, \bar{x})$).

Theorem 8.

Let $(z, z^*) \in \operatorname{gph} N_\Gamma$ and $(v, u) \in T_{\operatorname{gph} N_\Gamma}(z, z^*)$ be given. Then $N_{\operatorname{gph} N_\Gamma}((z, z^*); (v, u))$ is the union of all product sets $V^0 \times V$ associated with cones V of the form $F_1 - F_2$, where F_1, F_2 are closed faces of the critical cone $\mathcal{K}_\Gamma(z, z^*)$ satisfying

$$v \in F_2 \subset F_1 \subset [u]^\perp. \qquad (16)$$

Remark. Clearly for $(v, u) = (0, 0)$, (16) reduces to a result from [DR96].

Ad (v) Example:

$\Gamma = \mathbb{R}_+, (z, z^*) = (0, 0) \in \mathrm{gph}\, N_\Gamma$
$\mathcal{K}_\Gamma(z, z^*) = T_\Gamma(z) \cap [z^*]^\perp = \mathbb{R}_+,\ F_1 = \mathbb{R}_+, F_2 = \{0\}$

By virtue of [DR96],

$$\begin{aligned}
&N_{\mathrm{gph}\, N_\Gamma}(z, z^*)\\
&= (F_1 - F_1)^\circ \times (F_1 - F_1) \cup (F_1 - F_2)^\circ \times (F_1 - F_2) \cup (F_2 - F_2)^\circ \times (F_2 - F_2)\\
&= (\{0\} \times \mathbb{R}) \cup (\mathbb{R}_- \times \mathbb{R}_+) \cup (\mathbb{R} \times \{0\}).
\end{aligned}$$

For $(v, u) = (1, 0)$, by Theorem 8, one obtains

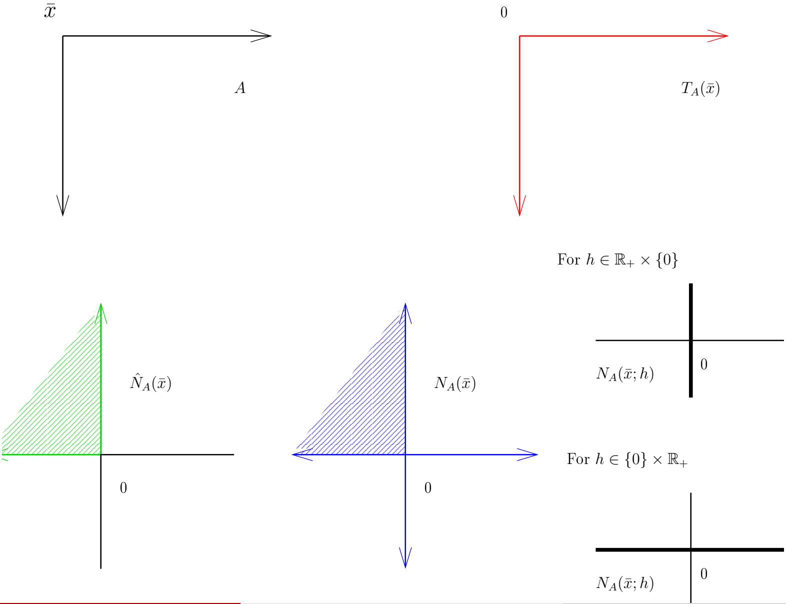
$$N_{\mathrm{gph}\, N_\Gamma}((z, z^*); (v, u)) = (F_1 - F_1)^\circ \times (F_1 - F_1) = \{0\} \times \mathbb{R},$$

because F_2 does not contain v.
Likewise for $(v, u) = (0, 1)$ one has

$$N_{\mathrm{gph}\, N_\Gamma}((z, z^*); (v, u)) = (F_2 - F_2)^\circ \times (F_2 - F_2) = \mathbb{R} \times \{0\},$$

because F_1 is not contained in $\{u\}^\perp$.

Ad (v) Example


$\bar{x}$
A
0
$T_A(\bar{x})$
For $h \in \mathbb{R}_+ \times \{0\}$
$N_A(\bar{x}; h)$
0
$\hat{N}_A(\bar{x})$
0
$N_A(\bar{x})$
0
For $h \in \{0\} \times \mathbb{R}_+$
$N_A(\bar{x}; h)$
0

Ad (v) Example

Consider the parameterized variational system.

$$0 \in H(p, x) + N_\Gamma(x)$$

with $p \in \mathbb{R}, x \in \mathbb{R}^2$,

$$H(p, x) = \begin{pmatrix} x_1 & - & p \\ -x_2 & + & x_2^2 \end{pmatrix}$$

and

$$\Gamma = \left\{ x \,\middle|\, \frac{1}{2} y_1 - y_2 \le 0, \frac{1}{2} y_1 + y_2 \le 0 \right\}.$$

Put $(\bar{p}, \bar{x}) = (0, 0)$. One has $K = T_\Gamma(0) = \Gamma$ and GE (14) attains the form

$$0 \in \begin{bmatrix} -v + u_1 \\ -u_2 \end{bmatrix} + N_\Gamma(u).$$

So the second assumption of Theorem 7 is fulfilled and one has to consider the following four pairs of directions:

A) $v \le 0,\ u_1 = v,\ u_2 = 0;$

B) $v \le 0,\ u_1 = \frac{4}{3} v,\ u_2 = -\frac{2}{3} v;$

C) $v \le 0,\ u_1 = \frac{4}{3} v,\ u_2 = \frac{2}{3} v;$

D) $v \ge 0,\ u_1 = u_2 = 0.$

Ad (v) Example

The relation at the left-hand side of (15) attains the form

$$\left(\begin{bmatrix} -b_1^* \\ b_2^* \end{bmatrix}, \begin{bmatrix} b_1^* \\ b_2^* \end{bmatrix}\right) \in N_{\mathrm{gph}\, N_\Gamma}\left((0,0); \left(u, \begin{bmatrix} v - u_1 \\ u_2 \end{bmatrix}\right)\right). \tag{17}$$

The faces of K are: $\mathcal{F}_1 = \{(0,0)\}, \mathcal{F}_2 = \mathbb{R}_+ \begin{bmatrix} -1 \\ 0.5 \end{bmatrix}, \mathcal{F}_3 = \mathbb{R}_+ \begin{bmatrix} -1 \\ -0.5 \end{bmatrix}$ and $\mathcal{F}_4 = K$.
A consecutive application of Theorem 8 in the cases A-D to (17) implies that in all of them $b^* = 0$. It follows that S has the Aubin property at $(0,0)$ (even without verification of the metric subregularity of the respective M).
Note that in this case we do not obtain the same conclusion by Theorems 1,4 or by the approach from [DQZ06]. $\triangle$

Consider the special case (3) with $Q(\cdot) = \widehat{N}_\Gamma(\cdot), \Gamma = q^{-1}(D)$ and assume that H is continuously differentiable, D is a convex polyhedron in $\mathbb{R}^s$ and $q : \mathbb{R}^m \to \mathbb{R}^s$ is a C^2-mapping. To simplify the analysis, we will impose the following standing assumption:

(A) $\bar{y}$ is *nondegenerate* for q with respect to D, i.e., one has the implication

$$\left. \begin{array}{l} \nabla q(\bar{x})^T \lambda = 0 \\ \lambda \in \mathrm{sp} N_D(q(\bar{x})) \end{array} \right\} \Rightarrow \lambda = 0. \tag{18}$$

Under (A) $\exists$ a neighborhood $\mathcal{N}$ of $\bar{x}$ such that for all $x \in \mathcal{N}$

$$\widehat{N}_\Gamma(x) = N_\Gamma(x) = (\nabla q(x))^T N_D(q(x))$$

and for each $\bar{x}^* \in \widehat{N}_\Gamma(\bar{x}) \exists$ a unique *Lagrange multiplier* $\lambda \in N_D(q(\bar{x}))$ such that

$$\bar{x}^* = (\nabla q(\bar{x}))^T \lambda.$$

Further we introduce the *Lagrangian*

$$\mathcal{L}(p, x, \lambda) := H(p, x) + (\nabla q(x))^T \lambda.$$

Ad (vi) Computation of $DM(\bar{p}, \bar{x}, 0)$

Proposition 1 ([HKO]).

Let $\bar{\lambda}$ be the (unique) Lagrange multiplier associated with $(\bar{p}, \bar{x})$, i.e.,

$$0 = \mathcal{L}(\bar{p}, \bar{x}, \bar{\lambda}), \bar{\lambda} \in N_D(q(\bar{x})). \tag{19}$$

Then for any $(v, u) \in \mathbb{R}^n \times \mathbb{R}^m$ one has

$$DM(\bar{p}, \bar{x}, 0)(v, u) = \nabla_p H(\bar{p}, \bar{x})v + \nabla_x H(\bar{p}, \bar{x})u + \nabla^2 \langle \bar{\lambda}, q \rangle (\bar{x})u + (\nabla q(\bar{x}))^T N_{\mathcal{C}}(\nabla q(\bar{x})u), \tag{20}$$

where $\mathcal{C} := \mathcal{K}_D(q(\bar{x}), \bar{\lambda}) = T_D(q(\bar{x})) \cap [\bar{\lambda}]^\perp$.

Note that

$$\nabla_x H(\bar{p}, \bar{x}) + \nabla^2 \langle \bar{\lambda}, q \rangle (\bar{x}) = \nabla_x \mathcal{L}(\bar{p}, \bar{x}, \bar{\lambda}).$$

Theorem 9.

Under the posed standing assumptions suppose that

- M is metrically subregular at $(\bar{p}, \bar{x}, 0)$;
- $\{u | 0 \in \nabla_p H(\bar{p}, \bar{x})v + \nabla_x \mathcal{L}(\bar{p}, \bar{x}, \bar{\lambda})u + (\nabla q(\bar{x}))^T N_{\mathcal{C}}(\nabla q(\bar{x})u)\} \neq \emptyset$ for all $v \in \mathbb{R}^n$;
- For every nonzero $(v, u) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$0 \in \nabla_p H(\bar{p}, \bar{x})v + \nabla_x \mathcal{L}(\bar{p}, \bar{x}, \bar{\lambda})u + (\nabla q(\bar{x}))^T N_{\mathcal{C}}(\nabla q(\bar{x})u)$$

  the implication

$$\left.\begin{array}{l} 0 \in (\nabla_x \mathcal{L}(\bar{p}, \bar{x}, \bar{\lambda}))^T b^* + \nabla q(\bar{x})^T D^* N_D((q(\bar{x}); \bar{\lambda}); (\nabla q(\bar{x})u, \mu))(\nabla q(\bar{x})b^*) \\ 0 = \nabla_p H(\bar{p}, \bar{x})v + \nabla_x \mathcal{L}(\bar{p}, \bar{x}, \bar{\lambda})u + (\nabla q(\bar{x}))^T \mu \\ \mu \in N_{\mathcal{C}}(\nabla q(\bar{x})u) \end{array}\right\} \Rightarrow$$

$$\Rightarrow b^* \in \ker(\nabla_p H(\bar{p}, \bar{x}))^T$$

  holds true.

Then S has the Aubin property around $(\bar{p}, \bar{x})$ and
$DS(\bar{p}, \bar{x})(v) = \{u | 0 \in \nabla_p H(\bar{p}, \bar{x})v + \nabla_x \mathcal{L}(\bar{p}, \bar{x}, \bar{\lambda})u + (\nabla q(\bar{x}))^T N_{\mathcal{C}}(\nabla q(\bar{x})u)\}$.

Consider the parameterized variational system.

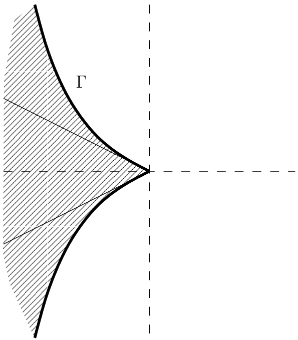
$$0 \in H(p, x) + N_\Gamma(x)$$

with $p \in \mathbb{R}, x \in \mathbb{R}^2$,

$$H(p, x) = \begin{pmatrix} x_1 & - & p \\ -x_2 & + & x_2^2 \end{pmatrix}$$

and

$$\Gamma = \left\{ x \,\middle|\, \frac{1}{2}y_1 - \frac{1}{2}y_1^2 - y_2 \leq 0, \frac{1}{2}y_1 - \frac{1}{2}y_1^2 + y_2 \leq 0 \right\}.$$



Ad (vii) Conclusion

1) The presented procedure has potential to be used in testing the Aubin property of solution maps to parameterized equilibria, governed by parameterized constraint and variational systems, where $\nabla_p G(\bar{p}, \bar{x}), \nabla_p H(\bar{p}, \bar{x})$ are not surjective, and

   - Λ, Γ are convex polyhedra (Theorem 8);
   - $\Gamma = \{x \in \mathbb{R}^m | q(x) \in D\}$, where D is a convex polyhedron or a possibly even a nonpolyhedral convex cone (e.g. Carthesian product of Lorentz cones).

   Note that even if Γ is polyhedral, the Aubin property of S around $(\bar{p}, \bar{x})$ *does not mean* that S has a single-valued and Lipschitzian localization because $\nabla_p H(\bar{x}, \bar{y})$ is not surjective.

2) Theorem 2 may well be applied also to nonsmooth equations where M is, for instance, a single-valued Lipschitzian mapping. In this way one can model parameterized complementarity and implicit complementarity problems.

3) The applied technique is based on a combination of primal-space and dual-space tools, namely the graphical derivatives and directional limiting coderivatives. This combinations proved its efficiency also in testing the calmness of S [GO15] and we plan to used also in the verification whether S has a single-valued Lipschitzian localization.

References

[DR96] Dontchev, Rockafellar: Characterization of strong regularity for VIs over polyhedral convex sets. SIOPT 6(1996), 1087-1105.

[DQZ06] Dontchev, Quincampoix, Zlateva: Aubin criterion for metric regularity. J. Convex Anal. 13(2006), 281-297.

[G11] Gfrerer: 1st and 2nd order characterizations of metric subregularity and calmness of constraint set mappings, SIOPT 21(2011), 1439-1474.

[GO15] Gfrerer, JVO: On Lipschitzian properties of implicit multifunction. NuMa Report No 2015-12, submitted to SIOPT.

[GM11] Ginchev, Mordukhovich: Directional subdifferentials and optimality conditions. Math. Res. Rep. 89(2011).

[HJO02] Henrion, Jourani, JVO: On the calmness of a class of multifunctions. SIOPT 12(2002), 603-618.

[HKO] Henrion, Kruger, JVO: Some remarks on stability of generalized equations. JOTA 159(2013), 681-697.

[M1] Mordukhovich: Variational Analysis and Generalized Differentiation I. Springer 2006.

THANK YOU